2.4

Real Zeros of Polynomial Functions

Long Division –

Division Algorithm for Polynomials :

Let *f*(*x*) and *d*(*x*) be polynomials with the degree of *f* greater than or equal to the degree of *d*, and *d*(*x*) $\neq 0$. Then there are unique polynomials *q(x)* and *r(x),* called the **quotient** and **remainder**, such that

$$f(x) = d(x) \bullet q(x) + r(x) \quad \text{(where } d(x) \text{ is the } \textbf{divisor}\text{)}$$

where either *r(x)* = 0 or the degree of *r* is less than degree of *d*.

If *r(x)*= 0 then *d(x)* divides evenly into *f(x).*

See Example 1 pg. 215

Remainder theorem: If a polynomial *f(x)* is divided by *x – k,* then the remainder *r = f(k).*

Example:

Find the remainder when $f(x) = 3x^2 + 7x - 20$ is divided by x – 2.

$$r = f(2) = 3(2)^2 + 7(2) - 20 = 12 + 14 - 20 = 6$$

Factor Theorem: A polynomial function *f(x)* has a factor *x - k* if and only if *f(k)* = 0.

See Example 2 pg. 216

Synthetic division:

See Example 3 pg. 218

Rational Zeros Theorem: Real zeros of polynomial functions are either rational zeros (rational numbers) or irrational zeros (irrational numbers).

Suppose *f* is a polynomial function of degree $n \geq 1$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0,$$

with every coefficient an integer and $a_0, \neq 0$. If $x = p/q$ is a rational zero of *f,* where *p* and *q* have no common integer factors other than 1, then

- *p* is an integer factor of the constant coefficient a_0, and
- *q* is an integer factor of the leading coefficient a_n.

Example:

Find all the possible rational zeros of $f(x) = 3x^3 + 4x^2 - 5x - 2$.

Potential rational zeros: $\dfrac{factors\ \ of\ \ -2}{factors\ \ of\ \ 3} = \dfrac{\pm 1, \pm 2}{\pm 1, \pm 3} = \pm 1, \pm 2, \pm \dfrac{1}{3}, \pm \dfrac{2}{3}$

Now how do we find **all** rational zeros of the polynomial?

Upper and Lower bound tests for Real Zeros:

Let f be a polynomial function of degree $n \geq 1$ with a positive leading coefficient. Suppose $f(x)$ is divided by $x - k$ using synthetic division.

- If $k \geq 0$ and every number in the last line is nonnegative (positive or zero), then k is an **upper bound** for the real zeros of f.
- If $k \leq 0$ and the numbers in the last line are alternately nonnegative and nonpositive, then k is a **lower bound** for the real zeros of f.

Example:

Prove that all of the real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$ lie in the interval [-2,5].

(Prove that -2 is a lower bound and 5 is an upper bound using synthetic division.)

Descartes's Rule of Signs:

Let P(x) be a polynomial with real coefficients written in standard form.

- The number of positive real roots of P(x) = 0 is either equal to the number of sign changes between consecutive coefficients of P(x) or is less than that by an even number.
- The number of negative real roots of P(x) = 0 is either equal to the number of sign changes between consecutive coefficients of P(-x) or is less than that by an even number.

  In both cases, count multiple roots according to their multiplicity.

Example using Descartes's Rule of Signs:

What does Descartes's Rule of Signs tell you about the real roots of

$2x^4 - x^3 + 3x^2 - 1 = 0$?

Answer:

$P(x) = 2x^4 - x^3 + 3x^2 - 1 = 0$ *has* 3 *sign changes between terms, therefore there are either* 3 *or* 1 *positive real roots.*

$P(-x) = 2x^4 + x^3 + 3x^2 - 1 = 0$ *has* 1 *sign change between terms, therefore there is* 1 *negative real root.*